LINEAR PROGRAMMING TECHNIQUES FOR REGRESSION ANALYSIS

BY
HARVEY M. WAGNER

TECHNICAL REPORT NO. 51

PREPARED UNDER CONTRACT N6onr-25133
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FOR
OFFICE OF NAVAL RESEARCH

DEPARTMENT OF ECONOMICS
STANFORD UNIVERSITY
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Harvey M. Wagner

Stanford University

1. Introduction

Karst [5] has recently suggested an iterative procedure "for finding a straight line of best fit to a set of two dimensional points such that the sum of the absolute values of the vertical deviations of the points from the line is a minimum." It is well known that the general $p + 1$ dimensional version of this problem may be exactly formulated as a linear programming model consisting of n equations, where n is the number of observations. By employing the fundamental dual theorem [1, 6, 8] in linear programming, we shall show how the problem can be solved by a p equation linear programming model with <u>bounded</u> variables [2, 3, 9]. Secondly we shall demonstrate how a <u>regular</u> $p + 1$ equation linear programming model can be utilized to find a line of best fit according to a Chebyshev criterion [4], i.e., a line (or hyperplane) which minimizes the maximum of the vertical deviations from the sample points.

2. Minimizing the Sum of Absolute Deviations

Let X denote an $n \times p$ dimensional matrix, where the columns consist of n observational measurements on p "independent" variables, and Y denote an n-dimensional column vector of measurements on the "dependent" variable.

We wish to find a p-dimensional column vector b such that

$$\begin{aligned} &Xb + Ie_1 - Ie_2 = Y \ , \ e_1, e_2 \geq 0 \\ &\text{minimize } E = (1\ 1\ \ldots\ 1) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \ , \end{aligned} \tag{1}$$

where I is an $n \times n$ identity matrix. We interpret e_1 and e_2 as n-dimensional column vectors of vertical deviations "below" and "above" the fitted line; i.e., $(e_1 + e_2)$ is the vector of absolute deviations between the fit Xb and Y (by the nature of the model, it is clear that the j-th components of e_1 and e_2 cannot both be strictly positive in an optimal solution). The solution to our problem yields the regression equation

$$(x_1\ x_2\ \ldots\ x_p) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} = xb = y. \tag{2}$$

Note that if we wish the left hand side of (2) to include a coefficient for the intercept of the y axis to be determined by the linear fit, then we can let $x_p \equiv 1$, and the p-th column of X be a vector of one's. We may force the fitted line to pass through some point, the usual example being the set of sample means, either by adding to (1) the linear restriction

$$(\bar{x}_1\ \ \bar{x}_2\ \ldots\ \bar{x}_p)\ b = \bar{y} \tag{3}$$

or by the usual least squares approach of subtracting each coordinate of the point, in our example the sample mean for each variable, from all the corresponding observations (including y) and then by fitting (1) without a y-intercept coefficient; the latter approach simply consists of shifting the origin of the axes in a p-dimensional space to the selected point, and then of fitting the line (hyperplane) through the new origin.

If it is desirable, the linear programming model (1) can be restricted further to permit only non-negative values for some or all of the components of b, and to force b to satisfy additional linear constraints. It is noteworthy that collinearity in X (even to the degree that two columns of X are identical) will not cause a failure in the algorithm for (1). One drawback of the model is evident: when the number of observations n is sizeable, (1) becomes computationally unwieldy.

We shall now transform (1) into a more manageable dual problem which will yield the optimal b as a byproduct. To start, we shall assume we have added to (1) the restriction $b \geq 0$. The fundamental dual relationship in linear programming [1, 6, 8] asserts a solution to (1) can be found by considering the linear programming model

$$X'd \leq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4a}$$

$$Id \leq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{4b}$$

$$-Id \leq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{4c}$$

$$\text{maximize } G = Y'd \quad , \tag{4d}$$

where X' is the transpose of X, Y' the transpose of Y, and d an n-dimensional column vector of "dual variables" which are unrestricted in sign (because (1) consists of a set of equations). Model (4), as it appears, is even a larger problem than (1), since it consists of p + 2n relations. To reduce the problem to a model in p equations and n bounded variables we let

$$(f) \equiv \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = (d) + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad . \tag{5}$$

Then (4) is equivalent to

$$X'f \leq X' \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad , \tag{6a}$$

$$0 \leq f_j \leq 2 \qquad j = 1, 2, \ldots, n \tag{6b}$$

$$\max \quad G^* = Y'f - Y' \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad . \tag{6c}$$

Upon appending a set of slack variables to (6a) and dropping the constant on the right side of (6c), we may solve (6) by one of the simplex algorithms for bounded variables [2, 3, 9]. If X and Y are deviations of sample values from their means, then the right hand side of (6a) is a vector of zero's, and the constant in (6c) is zero. Denoting the basis of the optimal solution of (6) by B (which may include slack vectors), and the associated coefficients in (6c) by r'_B, we have

$$b = (B^{-1})' r_B \quad . \tag{7}$$

No extra computations are needed to find (7). In the original simplex method b appears in the $(z_j - c_j)$ row of the final simplex tableau under the columns for the slack vectors [1, 8]; in the revised simplex method (7) is the "shadow price" vector for the optimal solution [7]. The optimal value of G^* is the minimized sum of absolute deviations.

When we drop our assumption that b be non-negative and allow the components of b to take on any sign, we modify (6) to

$$X'f = X' \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{6a'}$$

and introduce a set of artificial variables having an arbitrarily high cost to initiate one of the simplex algorithms. The optimal b remains (7), i.e., the shadow price vector in the revised simplex method or z_j of the final simplex tableau under the columns for the artificial vectors [1].

In summary, we can solve for b in (1) by applying a simplex algorithm for bounded variables to the p equation model (6). Although the mathematical manipulation underlying the transformation of problems appears involved, the computational procedure required to solve (6) is relatively straightforward, but somewhat laborious.

3. Minimizing the Maximum Absolute Deviation

The most bothersome aspect of the approach in the previous section is the requirement of a linear programming algorithm for bounded variables, as such techniques are (slightly) more difficult to perform than the standard simplex algorithm. We may eliminate the drawback if we are willing to accept a Chebyshev criterion for best fit. Our model in this case is to find a p dimensional column vector b such that

$$Xb - Y \leq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} e \qquad (8a)$$

$$-Xb + Y \leq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} e \qquad (8b)$$

$$\text{minimize} \quad e \geq 0 \quad . \qquad (8c)$$

Examination of (8) will reveal that e is the maximum absolute deviation. The equations (8) are reminiscent of a linear programming formulation for the minimax problem in two-person zero-sum games, and we shall use a similar approach for the solution. An equivalent expression for (8) is

$$\begin{pmatrix} -X & \vec{1} \\ X & \vec{1} \end{pmatrix} \begin{pmatrix} b \\ e \end{pmatrix} \geq \begin{pmatrix} -Y \\ Y \end{pmatrix} \tag{9}$$

$$\text{minimize} \quad e \geq 0 \ ,$$

where $\vec{1}$ is an n-dimensional column vector of one's. Our previous remarks concerning additional linear constraints on b apply here equally as well.

Assuming for the moment that we wish to impose the restriction $b \geq 0$, we convert the 2n equation model (9) into its dual form, which contains only $p + 1$ equations

$$(-X' \quad X') \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \leq (\vec{0}) \tag{10a}$$

$$(\vec{1}' \quad \vec{1}') \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \leq 1 \tag{10b}$$

$$h_1 , \ h_2 \geq 0 \ ,$$

$$\text{maximize} \quad M = (-Y' \quad Y') \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \ , \tag{10c}$$

where $\vec{0}$ is a p-dimensional column vector of zero's. The vectors h_1 and h_2 are n-dimensional columns; if a component of h_1 (h_2) is positive in the optimal solution of (10), then the maximum deviation occurs at the corresponding point or equation in (8), and this point will lie "below" ("above") the fitted line. Analogous to our result in (7),

$$\begin{pmatrix} b \\ e \end{pmatrix} = (B^{-1})'r_B \tag{11}$$

where B denotes the optimal basis for (10), and r_B' the coefficients in (10c) corresponding to the variables in B; and exactly as before, the solution (11) is a byproduct of the simplex method.

If we drop the assumption that b be non-negative, we need only change (10a) to equalities, and results analogous to those in the previous section continue to hold.

4. A Numerical Example

Karst [5] examines the following data

$$X' = [-12.5 \quad -8.5 \quad -6.5 \quad -3.5 \quad -2.5 \quad -1.5 \quad -0.5 \quad 2.5 \quad 4.5 \quad 8.5 \quad 8.5 \quad 11.5]$$

(12)

$$Y' = [-\ 8.4 \quad -5.4 \quad 3.6 \quad -2.4 \quad -4.4 \quad 1.6 \quad -0.4 \quad -0.4 \quad -2.4 \quad 3.6 \quad 5.6 \quad 9.6],$$

which comprise deviations of the original data about their sample means. He finds the least squares fit to be

$$y = .539\ x \quad , \tag{13}$$

and the fit for the minimized sum of absolute deviations to be

$$y = .659\ x \quad . \tag{14}$$

As we have argued, (14) can be obtained by (6), where specifically we would find

$$b = (B^{-1})'r_B = (1/8.5)\ 5.6 = .659 \quad . \tag{7'}$$

The solution by model (10) yields

$$B = \begin{pmatrix} -6.5 & 11.5 \\ 1 & 1 \end{pmatrix} , \quad r_B' = [3.6 \quad 9.6] \tag{15}$$

and consequently

$$\begin{pmatrix} b \\ e \end{pmatrix} = \begin{pmatrix} .333 \\ 5.767 \end{pmatrix} . \tag{11'}$$

That is, the Chebyshev fit is

$$y = .333\, x \quad ; \tag{16}$$

since the vectors in B correspond to variables in h_2 , the third and last sample point in (12) will lie above the fitted line and assume the maximum vertical deviation from it of 5.767.

REFERENCES

[1] Charnes, A., Cooper, W.W., and Henderson, A., An Introduction to Linear Programming, New York: Wiley, 1953.

[2] Charnes, A., and Lemke, C.E., "Computational Theory of Linear Programming: The Bounded Variables Problem," Graduate School of Indust. Admin., Carnegie Tech., 1954.

[3] Dantzig, G.B., "Upper Bounds, Secondary Constraints, and Block Triangularity in Linear Programming," Econometrica 23, 174-183, 1955.

[4] Hastings, C., Approximations for Digital Computers, Princeton: Princeton University Press, 1955.

[5] Karst, O.J., "Linear Curve Fitting Using Least Deviations," Journ. of the American Statistical Assoc., 53, 118-132, 1958.

[6] Tucker, A.W., "Dual Systems of Homogeneous Linear Relations," in H.W. Kuhn and A.W. Tucker, Linear Inequalities and Related Systems, Princeton: Princeton University Press, 1955, 3-18.

[7] Wagner, H.M., "A Comparison of the Original and Revised Simplex Methods," Operations Research 5, 361-9, 1957.

[8] Wagner, H.M., "A Practical Guide to the Dual Theorem," Operations Research 6, June 1958.

[9] Wagner, H.M., "The Dual Simplex Algorithm for Bounded Variables," ONR Report No. 50, Dept. of Economics, Stanford University, 1958; Naval Research Logistics Quarterly 5, September 1958.

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